Appendix: the Helmholtz Equation

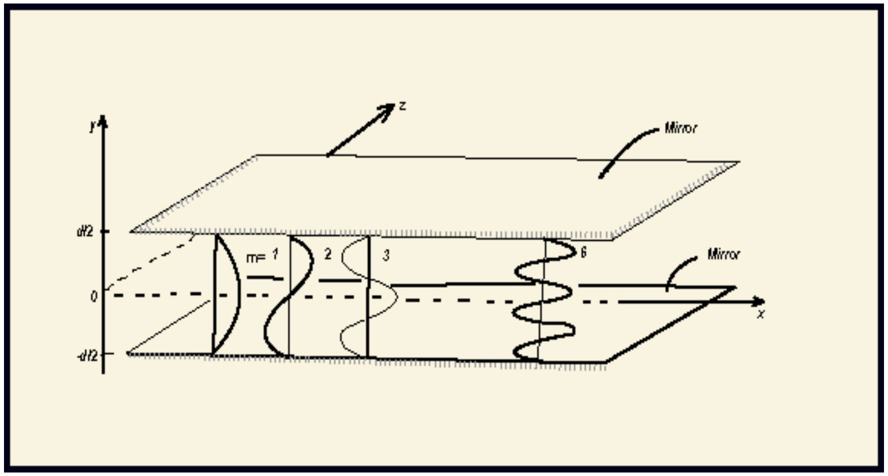
- Helmholtz Equation
- Finite Difference Method

Excerpts from MIT course #3.46 “Photonic Materials and Devices”

Guided Wave Optics

Confining Light in 2 out of 3 Dimensions…

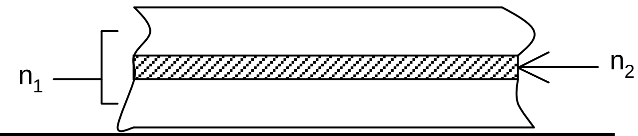
- What if we could *trap* a Traveling wave along one/two directions?
  - Standing wave: bounded by a different stiffness (permittivity ε)



1D trapping of wave : “Confinement”

- What might happen in the trap-free direction (z dimension)?

Guided Wave Optics



Waveguides

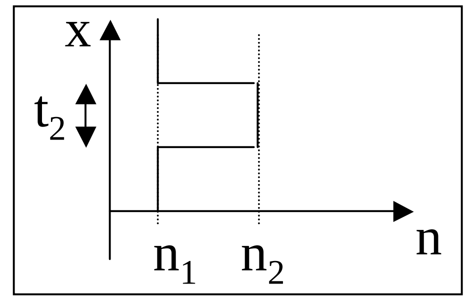
Maxwell's Equations (case: $\rho = 0, J_f = 0$)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu\vec{J}_f + \varepsilon\mu\frac{\partial\vec{E}}{\partial t} = \varepsilon\mu\frac{\partial\vec{E}}{\partial t}$$

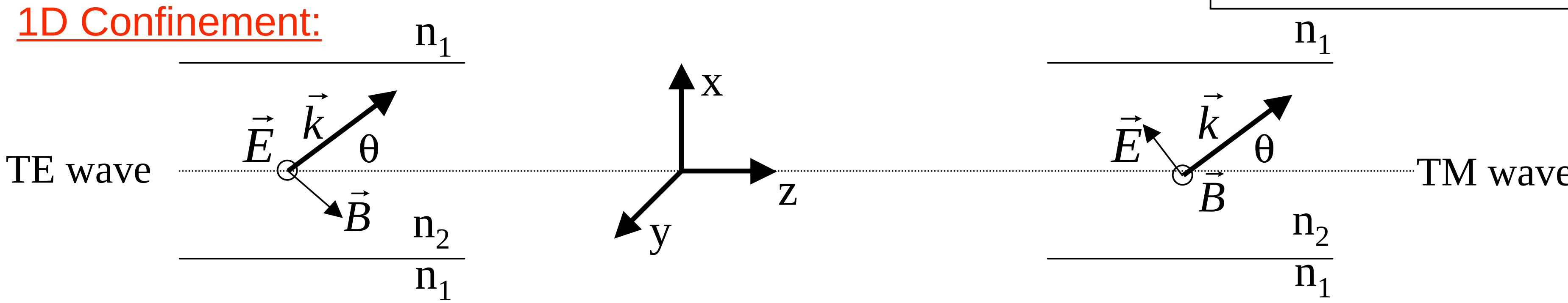
Identity: $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$

Traveling Wave Equation: $\nabla^2\vec{E} = \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = \frac{n^2}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$



- Additional constraint: no propagation in x & y directions
  - Is this even possible?
  - What can we manipulate: n_1, n_2, t_2 (assume $t_1 \to \infty$)

1D Confinement:



TE wave:
- $E_y \neq 0, E_x = 0, E_z = 0 \Rightarrow E_y = f(x)$
- $B_y = 0, B_x \neq 0, B_z \neq 0 \Rightarrow B_x, B_z = f(x)$

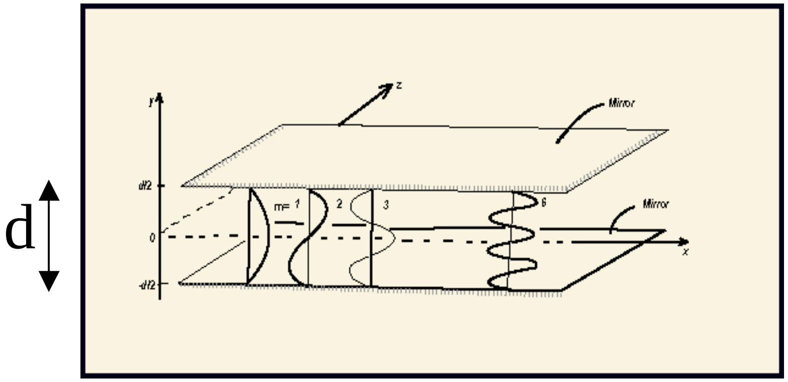
TM wave:
- $E_y = 0, E_x \neq 0, E_z \neq 0 \Rightarrow E_x, E_z = f(x)$
- $B_y \neq 0, B_x = 0, B_z = 0 \Rightarrow B_y = f(x)$

Guided Wave Optics

Waveguide Solution

(TE) Ansatz: $E_y = U(x) e^{-i(\beta z - \omega t)}$

$$\nabla^2 \vec{E} = \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$ Traveling Wave Eqn



Standing waves: no more progation

$$e^{-ikx+i\omega t} + e^{ikx+i\omega t} = 2\cos(kx) e^{i\omega t}$$

Helmholtz Equation:

$$\left[\frac{d^2}{dx^2} + k^2 - \beta^2\right] U(x) = 0, \;\; k = nk_0 = n\frac{\omega}{c}$$

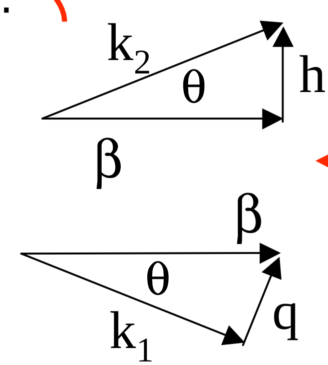
scalar eqn

$$U(x) = \begin{cases} A\sin(hx) + B\cos(hx) & |x| < \tfrac{1}{2}d \\ C\exp(-qx) & x > \tfrac{1}{2}d \\ D\exp(qx) & x < \tfrac{1}{2}d \end{cases}$$

$$h = \left[(n_2 k_0)^2 - \beta^2\right]^{1/2}, \; \beta < n_2 k_0$$

$$q = \left[\beta^2 - (n_1 k_0)^2\right]^{1/2}, \; \beta > n_1 k_0$$

- Ray Optic Intuition:
  - k_2 bounces around inside wg…
  - $k_1 < \beta < k_2$
- Effective Index: $\beta \equiv n_{eff} k_0$
  - $\Rightarrow$ $n_1 < n_{eff} < n_2$ $\quad (n_{eff} = n_2 \cos\theta)$ $\quad (\beta = n_2 k_0 \cos\theta)$



Excerpts: MIT course #3.46

Guided Wave Optics

Waveguide Solution

$$\text{(ii)}\ \vec{E}_{1\|} = \vec{E}_{2\|}$$ (continuity)

$$\text{(iv)}\ \frac{1}{\mu_1}\vec{B}_{1\|} = \frac{1}{\mu_2}\vec{B}_{2\|} \longrightarrow \frac{i}{\omega\mu}\frac{\partial E_x}{\partial y}\bigg|_1 = \frac{i}{\omega\mu}\frac{\partial E_x}{\partial y}\bigg|_2$$

(differentiability)

<u>4 b'dry conditions:</u>

$$A\sin(hd/2) + B\cos(hd/2) = C\exp(-qd/2)$$
$$hA\cos(hd/2) - hB\sin(hd/2) = -qC\exp(-qd/2)$$
$$-A\sin(hd/2) + B\cos(hd/2) = D\exp(-qd/2)$$
$$hA\cos(hd/2) + hB\sin(hd/2) = qD\exp(-qd/2)$$

$B = 0, h\tan(hd/2) = q$ — Odd solns

$A = 0, h\cot(hd/2) = -q$ — Even solns

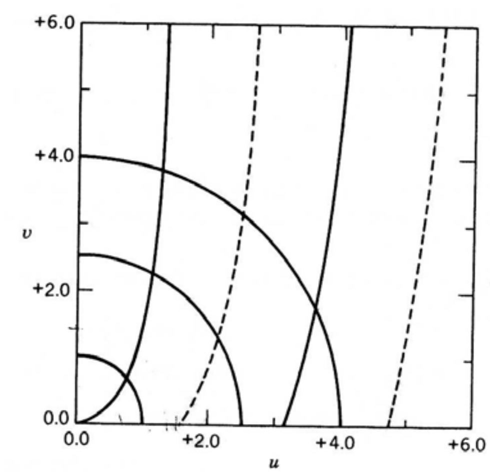
Define Unitless Parameters:

$$u \equiv hd/2, v \equiv qd/2$$

$$\begin{cases} \text{TE: } u\tan(u) = v, u\cot(u) = -v \\ \text{TM: } u\tan(u) = \left(\frac{n_2}{n_1}\right)^2 v, u\cot(u) = -\left(\frac{n_2}{n_1}\right)^2 v \end{cases}$$

Identity: $$u^2 + v^2 = (n_2^{\ 2} - n_1^{\ 2})\left(\frac{\omega d}{2c}\right)^2 = NA^2\left(\frac{\pi d}{\lambda_0}\right)^2 \equiv V^2$$

"V-parameter"



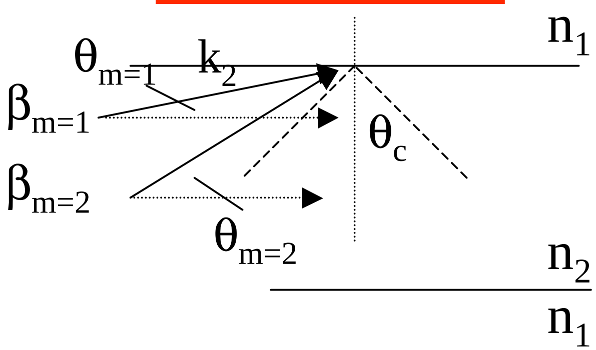
<u>Multiple solns</u>
- U(x): $U_m(x)$
- β: β_m
- $n_{eff} = n_{eff,m}$
- m=1,2,3,…

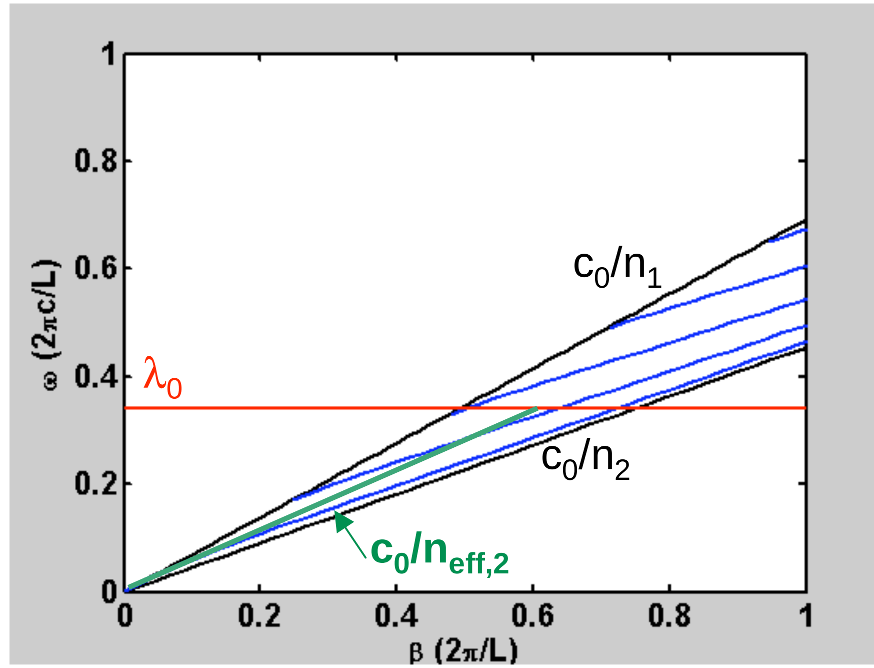
Guided Wave Optics

Waveguide Modes

- Multiple modes (m=1,2,3,…)
- Single-mode cut-off

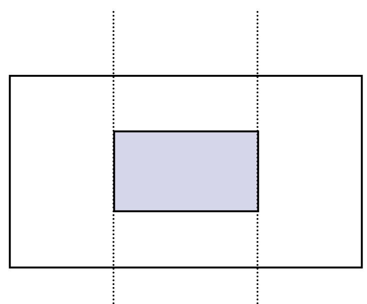
$$\beta_m = n_2 k_0 \cos\theta_m$$



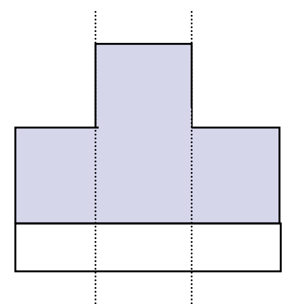


$$n_{eff,m} = n_2 \cos\theta_m$$

- How to deal with 2D confinement?
  - Effective index approach
  - Solve (up to 3) 1-D Eqns



(buried) strip wg



ridge wg

Excerpts: MIT course #3.46

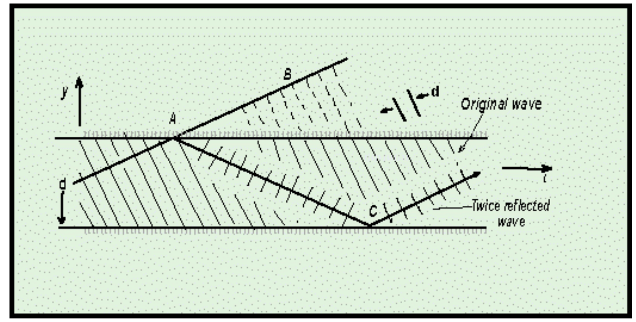
Guided Wave Optics

A Ray Optics and Wave Optics picture of Waveguides

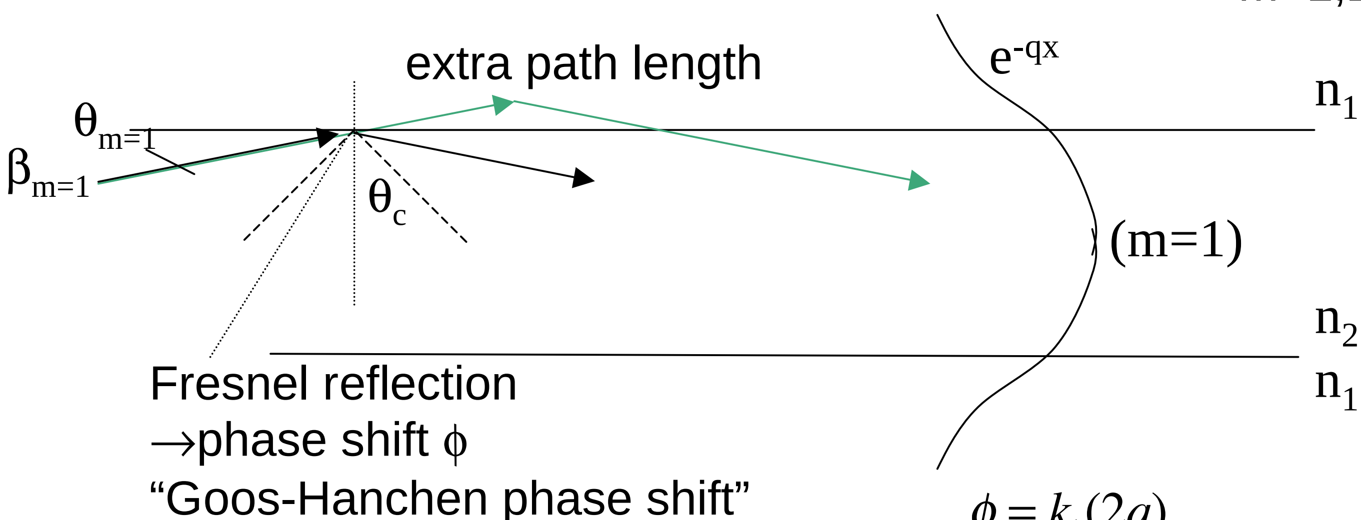
$$\beta_m = n_2 k_0 \cos\theta_m$$

- Alternative approach to transcendental eqn: **Total Internal Reflection**
  - Ray Optic + Fresnel Eqn: solve for $\cos\theta_m$ ('Fund. of Phot.' sec.7.2)
  - Bragg's Law: after two reflections, experience 2π-phase shift

- Ray Optic visualization



$m(\lambda_0/n_2)=2d\sin\theta_m$
$m=1,2,3,\ldots$



$$\phi = k_1(2q)$$

The Scalar Field Profile

Finite Difference Method

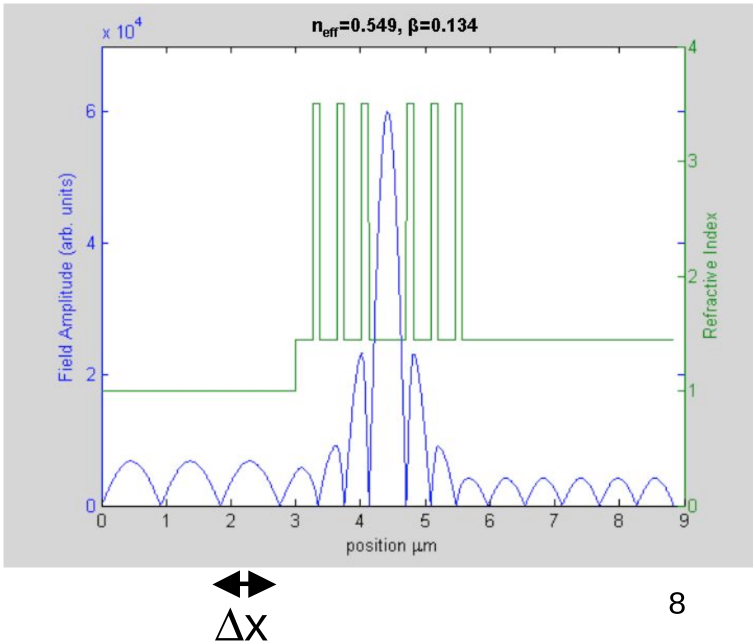
$$\frac{d^2}{dx^2}U_m(y)+k^2U_m(y)=\beta_m{}^2U_m(y) \rightarrow \frac{U_{i+1}}{(\Delta x)^2}+\left[k_0{}^2n_i{}^2-\frac{2}{(\Delta x)^2}\right]U_i+\frac{U_{i-1}}{(\Delta x)^2}=k_0{}^2n_{eff}{}^2U_m$$

$$\begin{bmatrix} n_1{}^2-2/(k_0\Delta x)^2 & 1/(k_0\Delta x)^2 & 0 & \cdots & 0 \\ 1/(k_0\Delta x)^2 & n_2{}^2-2/(k_0\Delta x)^2 & 1/(k_0\Delta x)^2 & 0 & \vdots \\ 0 & 1/(k_0\Delta x)^2 & \ddots & 1/(k_0\Delta x)^2 & 0 \\ \vdots & 0 & 1/(k_0\Delta x)^2 & n_{m-1}{}^2-2/(k_0\Delta x)^2 & 1/(k_0\Delta x)^2 \\ 0 & \cdots & 0 & 1/(k_0\Delta x)^2 & n_m{}^2-2/(k_0\Delta x)^2 \end{bmatrix}\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{m-1} \\ U_m \end{pmatrix}=n_{eff}{}^2\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{m-1} \\ U_m \end{pmatrix}$$

$$U_{1-1}=0, U_{m+1}=0$$

- Discretize x-axis
  - Helmholtz: differential → difference eqn
  - U(x) → U_i
  - $U_i = f(U_{i-1}, U_{i+1})$

(Applicable to TE-mode E-field profile)



Excerpts: MIT course #3.46