Quantum Mechanical Transmission with Absorption

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ABSTRACT

Transmission and reflection across a rectangular barrier at energies below and above barrier is the most commonly studied topic in non-relativistic quantum mechanics. However the subtle inter-relationship between the barrier problem and the corresponding well problem is not widely known, in particular when absorption is present. In this article we show that when a particle traverses an absorptive medium, at any given energy the absorption peaks for a particular value of absorption potential strength $W_0$. Similarly we study the corresponding cases when incident energy $E$ is increased keeping $W_0$ constant. Further, we show that for a given $E$ when $W_0$ is made very large absorption gradually decreases and reflection overtakes it and tends towards unity. We also study the case of transmission across a potential barrier and well and interpret physically the behavior of absorption, transmission and reflection.

1. Introduction

Transmission and reflection across a rectangular barrier at energies below and above barrier is the most commonly studied topic in non-relativistic quantum mechanics. However the subtle inter-relationship between the barrier problem and the corresponding well problem is not widely known, in particular when absorption is present. This problem has a number of physically interesting features, which throw much light on the underlying quantum mechanics. This is the subject matter of this article.

In quantum mechanics use of complex potentials with attractive imaginary part were...
introduced to study problems of collisions where there was a reduction of incident flux in the outgoing elastic channel due to absorption of the flux into non-elastic channels. Just as the problem of absorption of electromagnetic waves in a medium could be analyzed in terms of complex refractive index of the medium, it was found that problem of absorption of incident flux into reaction channels could be simulated in terms of a complex potential, appropriately called optical potential. Nuclear scattering study with such optical potentials is well known as nuclear optical model. Complex potential with attractive imaginary part attenuates the flux. However, attenuation is also found when flux with below barrier energy enters the real barrier affecting its transmission. That is, attenuation is not a particularly special feature of complex potential. Hence a careful study of reflection, absorption and transmission and their comparison with real potentials is desirable to highlight the subtleties involved in complex potential scattering in general and one dimensional transmission in particular. In this connection, it may be pointed out that in a previous work Lyngdoh et al have examined closely related problem of resonances in an absorptive well located in between barriers. This paper deals with several more features not dealt with in Ref. 5.

In Section 2 we give the mathematical preliminaries and deduce the relation between absorption, transmission and reflection. In Section 3 we study the transmission, reflection and absorption generated by an absorptive domain, absorptive well and barrier and examine their physical significance. Section 4 contains summary and conclusions.

2. Transmission, reflection and absorption

Let us consider the problem of transmission across a potential barrier given by

\[ V(x) = V_0 - i W_0, \quad 0 < x < a, \quad V_0 > 0, \quad W_0 > 0 \]

\[ = 0 \quad x > a \quad \text{or} \quad x < 0 \]

(1)

This potential represents an absorptive well when \( V_0 < 0, \) \( W_0 > 0. \)

The one dimensional time independent Schrödinger equation for this potential is

\[ \frac{d^2\Phi}{dx^2} + (k^2 - V(x))\Phi = 0 \]

(2)

where we have set \( 2m = 1, \) \( \hbar = 1 \) for convenience and energy \( E = k^2 \) and hence \( V(x) \) and \( E \) have dimension \( L^{-2}. \) We have used fm as unit of length in our numerical calculations. Defining

\[ \alpha^2 = k^2 - V_0 + i W_0 \]

(3)

the general solution \( \Phi \) can be written as

\[ \Phi(x) = \begin{cases} \exp(ikx) + B \exp(-ikx), & x < 0; \\ C \exp(ikx) + D \exp(-ikx), & 0 < x < a; \\ F \exp(ikx), & x > a; \end{cases} \]

(4)

where the incident wave amplitude is assumed to be unity. Imposing the continuity of \( \Phi \) and its derivative at \( x = 0 \) and \( x = a \) and solving the resulting algebraic equations leads to the following expressions for \( B \) and \( F: \)

\[ B = \frac{\left(1 - \frac{k^2}{\alpha^2}\right)\left(1 - e^{-2i\omega}\right)}{\left(1 + \frac{k^2}{\alpha^2}\right)e^{-2i\omega} - \left(1 - \frac{k^2}{\alpha^2}\right)} \]

(5)

\[ F = \frac{\left(1 - \frac{k^2}{\alpha^2}\right)\left(1 + \frac{k^2}{\alpha^2}\right)}{\left(1 - \frac{k^2}{\alpha^2}\right)e^{i(k\alpha)} - \left(1 + \frac{k^2}{\alpha^2}\right)e^{-i(k\alpha)}} \]

(6)

The corresponding reflection coefficient \( (R) \) and transmission coefficient \( (T) \) are

\[ R = |B|^2 \]

(7)
In the case of absorptive potential $R + T < 1$. Hence absorption coefficient $A = 1 - R - T$.

All our numerical analysis is based on the above expressions for different combinations of $V_0$, $W_0$ and $a$ parameters. It may be noted in the case of real potential $\alpha$ is real for real barrier if $E > V_0$ and purely imaginary when $E < V_0$. Similarly, $\alpha$ is real for all $E > 0$ for real well ($V_0 < 0$).

When the absorption strength $W_0 > 0$, $\alpha$ is complex for all $E > 0$.

Expression for $A$ can be deduced in a straightforward way. Using the Schrodinger equation (2) and its complex conjugate version we can show that

$$\left[ \frac{d\Phi^*}{dx} - \frac{d\Phi}{dx} \right] = 2iW_0 \int_0^a \Phi^* \Phi dx$$

In this we substitute appropriate expressions from (4) and then use (7) and (8). This gives us the relation

$$1 - R - T = \frac{W_0}{k} \int_0^a \Phi^* \Phi dx$$

Clearly, the right side of Eq.(10) signifies the absorption coefficient $A$ and satisfies the relation

$$R + T + A = 1$$

In the next sections we examine the relative importance of $R$, $T$ and $A$ in different domains using several examples.

Figure 1. Variation of coefficient of transmission and reflection across a rectangular barrier (indicated by $B$) ($V_0 = 5$, $a = 2$) and corresponding well ($W$) as a function of $E$. In this and rest of the figures potential strengths and $E$ are in fm$^{-2}$ units and $a$ in fm units.
3. Comparison of Transmission, Reflection and Absorption

3.1 Real barrier and the corresponding well

Let us first consider the transmission across a real barrier and the corresponding well. While explaining the tunneling problem, one normally emphasizes that the transmission across a real barrier when \( E < V_0 \) is a quantal phenomena which has no classical analogue. Implicitly, this assertion may give an impression to the students that when the barrier is replaced by a well, there would be full transmission without reflection. Hence, it is important to stress that in quantal one-dimension transmission problems, reflection and transmission are present for almost all potentials whether attractive or repulsive. This is because, in this case one is dealing with the problem of a wave propagation across different mediums. In order to demonstrate this we compare \( R \) and \( T \) both in the case of a real barrier and the corresponding well. In Figure 1 we show the variation of \( R \) and \( T \) as a function of \( E \) for a rectangular barrier and the corresponding well. As one can expect, the transmission across the well is substantially larger at below barrier energies than the corresponding barrier, but is less than unity. Similarly it may be noted that even a well can cause substantial reflection even though in most of the domain it is less than the corresponding value for the barrier. At above barrier energies transmission across the barrier becomes quite close to that of the well even though marginal difference persists. However, in the limit \( V_0/E \gg 1 \), \( T(B) \) and \( T(W) \) will be very close to each other. Figure 1 demonstrates all these salient features in a comprehensive way.

3.2 Absorption, Reflection and Transmission Across an Absorptive Domain

Now we examine the effect on absorption, reflection and transmission from a absorptive potential given by

\[
V(x) = -iW_0, \quad 0 < x < a
\]

\[
V(x) = 0, \quad x < 0 \text{ or } x > a
\]

This is useful in exploring the features generated by a purely absorptive domain when no real potential is present. We study \( R, T, A \) at a given energy as a function of \( W_0 \). Naively, one would expect that at a given \( E \) when \( W_0 \) increases indefinitely absorption would steadily dominate over reflection and transmission. In Figure 2 we show the variation of \( R, T \) and \( A \) as a function of \( W_0 \) for a fixed energy \( E = 3 \). It is clear from this figure that the transmission rapidly vanishes with the increase in \( W_0 \). However absorption shows an interesting behaviour. It rapidly raises with the increase in \( W_0 \) reaches a maximum and then starts slowly falling as \( W_0 \) is further increased. This means that at a given energy there is a critical strength of absorption \( W_0 \) when \( A \) is maximum and increasing it further does not lead to additional increase of \( A \). On the other hand reflection coefficient increases steadily with \( W_0 \). This gives rise to question whether \( R \) will approach unity as \( W_0 \) goes to infinity. The answer is in affirmative. By examining the behaviour of \( B \) as a function of \( W_0 \), we can see why \( R \to 1 \) as \( W_0 \) goes to infinity. In Eq.(5) when \( W_0 \to \infty \), \( k/\alpha \to 0 \) and hence \( B \to -1 \) and \( R \to 1 \) as absorption strength tends to \( \infty \). However this rise of \( R \) to unity is very slow as compared with a real barrier case. In the latter case when \( V_0 \) increases significantly above \( E \), \( R \) rapidly rises to unity and \( T \) exponentially falls. Thus, we may find that even though rapidly increasing \( W_0 \) tends to increase \( R \), because of slow decrease of \( A \) the rise of \( R \) to unity is also quite slow .In order to indicate the relative values of \( R,T \) and \( A \) as absorption strength \( W_0 \) becomes very much larger than \( E \) in Figure 3 we display the relative contribution of transmission \( T \), reflection \( R \) and
absorption ($A$) as a function of $Wo$ over a much wider range for a fixed energy. From this figure it is clear that at a critical $Wo$, $R$ exceeds $A$ and steadily increases towards unity.

Now we undertake the study of $R$, $T$ and $A$ as a function of energy $E$ for a fixed $Wo$.

In Figure 4 we study the variation of transmission ($T$), reflection ($R$) and absorption ($A$) with energy ($E$) for the absorptive potential with $Vo = 0$ and $Wo = 5$. Here also one finds that $A$ reaches a maximum when $E$ is in the neighborhood of $Wo$ and then gradually decreases. Thus Figure 3 and Figure 4 show that there is a critical $Wo$ for a given energy and critical $E$ for a given $Wo$ where absorption dominates. Here also there is a critical $E$ after which $T$ exceeds $A$. As one can expect, when $E$ becomes large $T$ predominates over $A$ and tends slowly towards unity and $R$ becomes negligible. Thus, in this case $T$ plays the role that $R$ played in Figure 3.

In the light of the results displayed in Figures 2-4, it is interesting to see the correlation between $E$ and $Wo$, which correspond to the maximum of absorption $A$. That is, we wish to examine the variation of $Wo(E)$ as a function of $E$ where $Wo(E)$ specifies the value of $Wo$ which gives absorption maximum at $E$. We demonstrate this correlation in Figure 5 for two different ranges of the absorptive potential. We find that $Wo(E)$ varies with respect to $E$ monotonically but in a non-linear way.

3.3 Comparison of Absorptive Barrier and Absorptive Well

Now we study the relative role of absorption ($A$) in an absorptive barrier ($V(x) = Vo − iWo$, $0 < x < a$) versus absorptive well ($V(x) = −Vo − iWo$, $0 < x < a$) with $Vo > 0$. The results are shown in Figure 6. We have taken the parameters $Vo = 5$, $Wo = 5$ and $a = 1.5$. The symbol $AW$ and $AB$ in this figure refer to absorptive well and absorptive barrier respectively. The variation of transmission $T(AW)$ across an absorptive barrier and the corresponding $T(AB)$ as a function of $E$ are more or less as expected and transmission $T(AW) > T(AB)$. Similarly the reflections $R(AW)$ and $R(AB)$ also show the variations as one anticipates. However more interesting behavior is the variation of absorption $A$ with energy. In the case of absorptive well the peak in absorption $A(AW)$ occurs at much lower energy than the corresponding absorptive barrier case. Further one finds that in most of the above barrier region of energy $A(AB)$ is more predominant than $A(AW)$. This can be explained as follows:

Consider the case of absorptive well. Since total energy $E$ is conserved, in the well domain $0 < x < a$, the kinetic energy increases. Due to this the particle (wave) system spends relatively less time in the well domain resulting in relatively less absorption as a function of energy. The argument works other way in the case of absorptive barrier at above barrier energies. In this case, in the barrier domain $0 < x < a$ kinetic energy decreases. This facilitates the system to spend more time in the barrier region facilitating higher absorption. Due to this as $E$ increases above $Vo$ one has $A(AB) > A(AW)$. Similarly, in most of the range having $E < Vo$, $A(AW) > A(AB)$ because, in this case it is more difficult for the system to penetrate into the barrier region than into the well region. Based on this physical picture one can similarly interpret the variation of pairs of variables $T(AW)$, $T(AB)$ and $R(AW)$, $R(AB)$.

4. Summary and Conclusions

In this paper we first studied the variation with energy the transmission and the reflection from a rectangular barrier and the corresponding well with out the presence of absorption. Then, we studied the variation of transmission, reflection and absorption generated by a purely
absorptive domain of width $a$ and strength $W_0$. We found that for a given energy absorption reaches a maximum at a particular $W_0$ and then gradually approaches zero as $W_0 \to \infty$; on the other hand reflection steadily approaches unity as $W_0 \to \infty$. The value of $W_0(E)$ at which absorption maximum occurs for a given energy $E$ increases steadily when $E$ is increased. Further we studied the variation of absorption, transmission and reflection caused by the absorptive barrier of finite height and the corresponding well and interpreted their variation with energy based on physical arguments. This article together with the Ref. 5 provide a comprehensive study of absorption, transmission, reflection and resonance phenomena in one dimensional quantum mechanical transmission problems.

![Graph](image)

Figure 2. Variation of co-efficient of absorption ($A$), transmission ($T$) and reflection ($R$) for an absorptive potential as a function of absorption strength ($W_0$) for a fixed $E = 3$. 
Figure 3. Variation of coefficient of absorption ($A$), transmission ($T$) and reflection ($R$) as a function of absorption strength ($Wo$) for a given energy $E=2$ indicating the gradual dominance of $R$ for large $Wo$. At $E=2$, $T$ becomes practically zero when $Wo > 9$.

Figure 4. Variation of co-efficient of transmission ($T$), reflection ($R$) and absorption ($A$), as a function of Energy ($E$) for an absorptive potential with parameters ($Vo = 0$, $Wo = 5$, $a = 1$).
Figure 5. Variation of the absorption strength \((Wo(E))\) where absorption peak occurs as a function of \(E\) in two different cases of absorptive potential \((Vo = 0, a = 1; Vo = 0, a = 2)\)

Figure 6: Comparative behaviour of coefficient of transmission \((T)\), reflection \((R)\) and absorption \((A)\) as a function of energy \((E)\) in the case of an absorptive barrier \((Vo = 5, Wo = 5, a = 1.5)\) and absorptive well \((Vo = -5, Wo = 5, a = 1.5)\)
References


