The Alleged "black hole" in Newtonian Gravity and the Event Horizon in Einstein's General Relativity when a Repulsive Force and a Vacuum Energy are Included

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ABSTRACT
In Newtonian gravity the horizon of invisible massive bodies are found from the criterion that light particles leaving the surface of a matter sphere will fail to travel to arbitrarily great distances. In Einstein’s geometric theory for gravitation a massive body may curve space and time in its neighborhood such that this body is surrounded with an event horizon. Neither particles nor light can leave this body and move outwards. It is a curious coincidence that these very different considerations and concepts yield just the same answer for the position of the horizon. To examine this coincidence we generalize Newton's law for gravitation to include a repulsive force. We also generalize the relativistic case to include vacuum energy. We find that the new criteria to have horizons are now different and this is also the case for the positions of the horizons. Our conclusion is that the Newtonian “black hole” is just a misnomer, and the coincidence does not point to any deeper connection between Newton’s gravity and general relativity.
1 Introduction

Almost any textbook on general relativity contains a chapter on relativity's most strange prediction, i.e. the existence of black holes. In that case the geometry of space and time outside a massive object is so strongly curved that there exists a fictive surface denoted by the event horizon from which not even light can escape.

In many popular texts it is even claimed that these peculiar objects are contained within Newton's theory of gravitation. If gravity is attractive and if light is affected by this force like all other forms of matter, there may exist massive radiating objects which are invisible for observers far away. The procedure applied to obtain this conclusion is just the well known calculation used to find the escape velocity when we want to decide if a planet is able to keep its atmosphere.

It is quite strange that the position of the horizon which light particles from the radiating body cannot penetrate, and the position of the event horizon found from a relativistic consideration, are just the same. One might even speculate whether this coincidence points to some deep connection between Einstein's general relativity and Newton's model for gravitation.

To the best of our knowledge an elementary and pedagogical discussion of this peculiar situation has never been given. The motivation for this paper is mostly educational. The calculations and argumentation should be easy to follow for students who are somewhat familiar with classical mechanics and are taking their introductory course in general relativity. The challenge concerning general relativity is that we cannot imagine curved 3-dimensional space. However, mathematics is superior since it gives us an understanding of what is going on even if we cannot make a proper picture of the situation.

We emphasize the two very different ways of doing physics. First we use Newton's classical theory for gravitation and thereafter we employ Einstein’s theory for gravitation, i.e. general relativity. It is our aim and hope that students will benefit from looking at the same problem from two different viewpoints and attacking the problem with different tools. A comparison of the solutions and methods should contribute to a better understanding of general relativity's superior way of treating problems concerning gravitation. We also firmly believe that physics teachers and the general physicist will profit from reading the paper.

To examine if the horizon obtained from Newton's theory has anything to do with the event horizon found using general relativity we generalize the two different theories in equivalent ways. Concerning Newton's gravitational law we simply add a repulsive force. Concerning Einstein’s theory we include repulsive vacuum energy. It would be quite surprising if the two theories still give the same answer and the coincidence remains. We arrive at the result we suspected: With our generalization the criteria to have horizons are not only more complicated, they are different. This conclusion is also valid for the eventual positions of the horizons.

2 Force or Curvature

Newtonian gravity is conceptually quite different from Einstein’s theory for gravitation, the general theory of relativity (GR). The concept of a force is foreign to general relativity. Gravitational forces are replaced by the curvature of the fabric of space and time. This curvature tells matter how to move in space and time. This distinction should be important concerning the language we use when we interpret what Einstein’s theory tells.

In our opinion this difference is very clearly disclosed comparing the Newtonian equation
of hydrostatic equilibrium for an ordinary star with the Tolman-Oppenheimer-Volkov (TOV) equation for a relativistic star. From Newton's second law and his law for gravitation we obtain:

$$\frac{dp}{dr} = -\frac{G \rho M(r)}{r^2}$$ (1)

where $p$ is the pressure, $r$ is the distance from the center of the star, $M$ is the mass contained within the sphere with radius $r$ and $G$ is Newton's gravitational constant. This equation tells that pressure is the force which via the pressure gradient resists the attractive gravitational forces trying to collapse the star.

On the other hand, for a spherically symmetric and static perfect fluid Einstein's field equations yield the TOV-equation.

$$\frac{dp}{dr} = \frac{1}{2} \left(\frac{G M}{c^2 r} \right) c^2 \frac{dr}{c^2 r} - \frac{1}{2} \frac{G M}{c^2 r}$$ (2)

where $c$ is the speed of light in vacuum. Comparing the equation of hydrostatic equilibrium and the TOV-equation it is seen that a static relativistic star demands a larger pressure gradient than a Newtonian star. Moreover, the appearance of the pressure on the right hand side of the TOV-equation is quite counterintuitive and tells that a positive pressure adds to the tendency matter has to contract under gravitation.

However, the TOV-equation further demonstrates the paradoxical and ambiguous double role played by the pressure. If we increase the pressure we also increase the pressure gradient which resists contraction of the star. But Newton's theory is inferior to Einstein's general relativity concerning gravitation. Hence, we should abandon the Newtonian concept of force for a relativistic star. The TOV-equation is just a relation which necessarily must be fulfilled to have a static relativistic star. If this relation is not valid the star will contract or expand.

### 3 The Black Hole

If the mass of the relativistic star is large this contraction will end as a catastrophic collapse where the surface of the star will contract within its own Schwarzschild surface. If that is the case there will be formed a so-called black hole in space-time. The line element outside a spherically symmetric static star was found by Karl Schwarzschild and is given by:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2$$ (3)

where $\theta$ and $\phi$ are the usual angular coordinates. The young dead Jörg Tofte Jebsen was the first to show that this metric is also valid outside a non-static star.

The Schwarzschild radius is given by:

$$r_s = r(Schwarzschild) = \frac{2GM}{c^2}$$ (4)

Inside the Schwarzschild surface the coordinate $r$ is a time marker and the coordinate $t$ is a radial marker. The important message from GR is that in the region outside the star and within the Schwarzschild surface a particle cannot remain at a constant value of the coordinate $r$. This means that all light cones in this region point inwards. Hence, neither massive particles nor photons can leave the surface of the star when this surface has collapsed within the Schwarzschild surface. This surface is an event horizon. No information from the region inside this horizon can move outwards.
4 The Historical “Black Hole”

The story goes that already in 1784 the English scientist John Michell obtained this Schwarzschild limit. He considered a particle falling from infinity towards a spherical globe with the same matter density as the Sun, but with a radius 500 times larger. The particle would reach the surface of this celestial body with a speed larger than the speed of light. Hence, a light particle leaving the surface of this heavenly body would not reach infinity. The gravitational force would compel the particle to return to its point of departure. Michell claimed it would be impossible to observe this celestial body directly.

However, it might disclose its existence if there were luminous satellites orbiting the central globe. The movement of the moons would display the unseen gravitational source. This reminds us of the argument now employed by the astronomers to demonstrate the existence of the so-called dark matter. The famous astronomer William Herschel even proposed that some of the glowing nebulae he observed were light captured by gravity.

The eminent mathematician, physicist and astronomer Pierre-Simon Laplace independently presented the equation corresponding to the total energy of the particle to be conserved. The case is settled and done with if the light particle can reach infinity with a finite speed. In 1796 Laplace pointed out without proof that a shining body with the same density as the Earth, but with a radius 250 times the radius of the Sun would prevent its light reaching observers very far away. Hence, the largest bodies in the universe would remain invisible. Laplace submitted the proper mathematical evidence in 1798.

The principle of energy conservation for a light particle with mass \( m \) and speed \( c \) leaving the surface of a matter sphere with radius \( R \) and mass \( M \) reads:

\[
\frac{1}{2} m v^2 = \frac{G M m}{r} + \frac{1}{2} m c^2 - \frac{G M m}{R} \quad (5)
\]

where \( v \) is the speed of the particle at the moment when its distance from the center of the gravitational source is \( r \). Hence, Laplace’s conclusion was that a horizon will exist if and only if:

\[
R < \frac{2G M}{c^2} \quad (6)
\]

i.e. the surface of the sphere must be within its Schwarzschild surface. This criterion is thus the same condition which GR demands for an event horizon to exist outside a spherically symmetric matter distribution. However, it should be remarked that the position of the event horizon is given by

\[
r_e = \frac{2GM}{c^2} \quad (7)
\]

which is independent of the size of the gravitational source. On the other hand the position \( r_h \) of the horizon for the Laplacian “black hole” is given by

\[
r_h = R \left( 1 - \frac{R}{r_h} \right)^{-1} \quad (8)
\]

Hence, it is seen that if a horizon exists we cannot decide if this horizon is outside or inside the event horizon. If it is outside we have \( R > \frac{1}{2} r_h \), and if it is inside we have \( R < \frac{1}{2} r_h \). Moreover, it is interesting to notice that

\[
\frac{d r_h}{dR} \left( 1 - \frac{R}{r_h} \right)^2 > 0 \quad (9)
\]

Hence, if the matter sphere contracts the horizon also move inwards while the event horizon remains at the original position. This conclusion is what we intuitively expect. It will be more difficult for the light particle to move...
away from the gravitational source when the negative gravitational potential increases its absolute value.

Already in 1808 Laplace removed every trace of this problem from his comprehensive treatise *Exposition of the System of the World*. Certainly this omission was due to Thomas Young’s discovery of the interference of light. This revelation convinced scientists that light was a kind of wave phenomenon. Both Michell and Laplace had accepted Newton’s model for light: Light is composed of tiny spheres moving rapidly through space. Then it was obvious that these insignificant bodies had mass and would be influenced by gravitational forces. Johann Georg von Soldner applying Newton’s theory even anticipated Einstein’s prediction that a light ray from a star passing close to the Sun will have its orbit distorted. The change of the star's position should be 0.84 arch seconds compared with the location when the Sun did not interfere with the light ray from the star.

When Newton’s particle model for light fell out of favour there was no longer any reason to suspect that light would be slowed down when climbing out of the grip of the gravitational field of the massive body. However, the fact remains that two very different considerations yield just the same criterion. In the Newtonian model the massive light particles can always leave the surface of the “black” star. There may exist a horizon the particles cannot penetrate, and they thus fail to travel to arbitrary large distances. However, an observer can always obtain information from the star going inside the horizon, and there is no problem concerning his return if traveling with a proper rocket.

Einstein’s general relativity tells a very different story. If the surface of the star has contracted inside its event horizon the curvature of space-time is so strong that a photon cannot leave the star. A curious observer who really wants to know what happens to the contracting star must go inside the event horizon. He will not obtain any information before he reaches the surface of the collapsing star. Then he is not able to send back information. But worse, he is now doomed to follow the star during the final stages of its catastrophic collapse. The conclusion is that the black hole concept does not exist in Newtonian theory. It is a phenomenon belonging to GR alone.

5 The Generalized Schwarzschild Black Hole

We shall now generalize our discussion and examine what happens to these horizons when a positive cosmological constant is included. This term was inserted by Einstein into his field equations to obtain a static universe. He was also motivated by his hope to satisfy Mach's principle, i.e. to explain the inertia of bodies and the existence of the distinguished inertial reference frames. Einstein discarded the cosmological constant as the biggest blunder in his life when Hubble's observation of the redshift of the light from the distant galaxies showed that non-static universe models took precedence over his static model.

However, nowadays the cosmological constant has become fashionable. It seems that the recent cosmological observations cannot be explained if this term is denied its presence in the gravitational field equations. But now this constant is given a better physical interpretation. It is a strange, but acceptable form of energy belonging to pure vacuum. Hence, the inclusion of this term has really nothing with cosmology to do, even if it historically was introduced into general relativity via that topic.

When the cosmological constant $\Lambda$ is included, the line element outside a spherically symmetric mass distribution is given by

\[ ds^2 = -c^2 dt^2 + a^2(r)^2 (\frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2) \]
\[ ds^2 = \left(1 - \frac{2GM}{rc^2} - \frac{1}{3} \Lambda r^2 \right) c^2 dr^2 - \frac{dr^2}{\left(1 - \frac{2GM}{rc^2} - \frac{1}{4} \Lambda r^2 \right)} - rd\theta^2 - r^2 \sin^2 \theta d\phi^2 \] (10)

To disclose the event horizon we can repeat the argument we employed examining the Schwarzschild metric. However, we want to mention that Adler et al.\(^4\) have disclosed a beautiful simple general criterion with an invariant meaning to examine if particles can pass a hyper-surface in one direction only or if physical objects can pass this surface in both directions.

Let the hypersurface \( S \) be given by

\[ u(x^\mu) = \text{constant} \] (11)

and define the vector \( n_\alpha \) by

\[ n_\alpha = \frac{\partial u}{\partial x^\alpha} \] (12)

The vector \( n_\alpha \) is normal to \( S \) since we have

\[ n_\alpha dx^\alpha = \frac{\partial u}{\partial x^\alpha} dx^\alpha = du = 0 \] (13)

for any vector \( dx^\alpha \) contained in \( S \). Particles can pass \( S \) in one direction only if

\[ n_\alpha n^\alpha >= 0 \] (14)

If \( n_\alpha n^\alpha < 0 \) particles can pass \( S \) in either direction. The case \( n_\alpha n^\alpha = 0 \) is the critical case. Here the one-way behavior begins, and Adler et al. denote this as a one-way membrane. Hence, this is where we find the event horizon. For our line element (10) a particle can pass the surface \( r = \text{constant} \) in one direction only if and only if

\[ 1 - \frac{2GM}{rc^2} - \frac{1}{3} \Lambda r^2 < 0 \] (15)

We also have

\[ \frac{d}{dr} \left(1 - \frac{2GM}{rc^2} - \frac{1}{3} \Lambda r^2 \right) = \frac{2GM}{c^2 r^2} - \frac{2}{3} \Lambda r \] (16)

Hence, the left hand side of (15) takes its maximal value \( 1 - \Lambda \left(\frac{3GM}{c r^3}\right)^{2/3} \) when \( r = \left(\frac{3GM}{\Lambda c}\right)^{1/3} \). Our conclusion is that for the geometry represented by the generalized Schwarzschild metric there may exist a one-way membrane if and only if

\[ \frac{1}{\sqrt{\Lambda}} > \frac{3GM}{c^3} \] (17)

To find the position of this event horizon we must solve the following equation

\[ r^3 - \frac{3}{\Lambda} r + \frac{6GM}{\Lambda c^2} = 0 \] (18)

When condition (17) is fulfilled equation (18) has one negative and two positive roots \( r_1 \) and \( r_2 \). With \( \Phi \) defined by

\[ \cos \Phi = -\frac{3GM}{c^2} \sqrt{\Lambda} \] (19)

we obtain

\[ r_1 = \frac{2}{\sqrt{\Lambda}} \cos \left(\frac{\Phi}{3} \right) \] (20)

and

\[ r_2 = -\frac{2}{\sqrt{\Lambda}} \cos \left(\frac{\Phi}{3} + 60^\circ\right) \] (21)

Remembering that \( \cos \Phi < 0 \) it must be the case that we can write \( \Phi/3 = 30^\circ + A \) or \( \Phi/3 = 90^\circ - A \) where \( 0^\circ < A < 30^\circ \). The quantity \( A \) is found using equation (19).

It is then easily seen that the smallest positive root \( R_1 \) of equation (18) is given by

\[ R_1 = \frac{2}{\sqrt{\Lambda}} \sin A \] (22)
and the largest root $R_2$ reads

$$R_2 = \frac{1}{\sqrt{\Lambda}}(\sqrt{3} \cos A - \sin A) \quad (23)$$

However, we must also demand that the generalized Schwarzschild geometry is valid in the region we consider. Hence, even if the necessary condition (17) is fulfilled several different cases can exist

1. $R < R_1$
   In this case we have two different one-way membranes. In the region $R < r < R_1$ particles and light can only move inwards. Here the effect of attractive gravitation dominates. In the region $R_1 < r < R_2$ physical objects can move in both directions. In the region outside the second one-way membrane, i.e. $r > R_2$, the repulsive cosmological term dominates and particles and light can only move outwards.

2. $R_1 < R_1 < R_2$
   In this case we have an outer one-way membrane only. In the region $R < r < R_2$ physical bodies can move both ways, and in the region $r > R_2$ material particles and light can only move outwards.

3. $R > R_2$
   In this case the effect of the repulsive cosmological term forces matter and light to move outwards.

Our conclusion is that for the generalized Schwarzschild metric an event horizon (one-way membrane) exists if and only if the following two conditions are both fulfilled:

$$\frac{2GM}{r} > \frac{\Lambda r}{3}$$

and $R < R_2$ where $R_2$ is the largest solution of the equation

$$r^3 - \frac{1}{\Lambda} r + \frac{6GM}{\Lambda^2} = 0.$$

6 Motivation for Choice of Generalized Newtonian Gravity

We shall now generalize Newton's gravitational theory to include a term corresponding to the so-called cosmological constant in Einstein's theory for gravitation. Both Einstein’s field equations and Newton’s inverse square law for gravitational effects are fundamental relations, and we cannot state any reasons why nature obeys these principles. The basic laws as we know them certainly represent intelligent and happy guesswork. But what is simple and beautiful no doubt was a strong guide to find the basic law for an acceptable theory. However, only nature will decide if the proposed framework is also true. Hence, when we want to generalize Newton’s theory, it is not possible to avoid doing it in an ad hoc fashion. Basically the answer is more or less just a matter of philosophical taste. Now we shall explain what is the motivation for our choice.

When Isaac Newton was working on his theory for gravitation he was aware that there was a gap in his argument concerning the motion of planets around the sun. He had assumed that the gravitational force exerted by a sphere is exactly the same as that exerted by an idealized point of mass located at its center. This is called the spherical property, and Newton in fact delayed the publication of his law of gravitation for about 20 years. However, he eventually found that he was correct after all, and the momentous Mathematical Principles of Natural Philosophy was published in 1687. Here we want to mention that now the reading of Newton's Principia is greatly facilitated by S. Chandrasekhar’s excellent book Newton’s Principia for the Common Reader.

But along the way Newton also discovered that the spherical property is fulfilled for one more case. It is also fulfilled for the so-called harmonic law, i.e. if the force between spherical masses is proportional with their...
distance apart. However, he could not accept that the force of gravity should increase as the distance between the masses grew in space and discarded this kind of force.

More than hundred years later “the French Newton”, Pierre-Simon Laplace, showed that the only possible force that has the spherical property must be the sum of 2 terms, an inverse square term and a term proportional with the distance between the objects. Laplace included the discussion in his magnificent text *Celestial Mechanics*. And like Newton before him he ignored the strange harmonic law term in his further discussion of gravitational phenomena.

The strange term remained forgotten until Einstein in 1915 established his geometrical theory for gravitation, the general theory of relativity. But when in 1917 he wanted to apply his new theory on a cosmological scale he discovered to his disappointment that no static cosmological solution of his original field equations exists. He then reintroduced the harmonic law term via a so-called cosmological constant. Choosing a specific value for this constant the total gravitational force in the universe was zero. Later it was shown by Georges Lemaitre that this term could not save the static model. Einstein’s universe was not stable. If a small disturbance expanded the universe somewhat, this expansion would continue forever. On the other hand, if a disturbance contracted the universe somewhat the world would experience a catastrophic collapse. When in 1929 Edwin Hubble made his remarkable discovery of the linear relation between the redshift of the light received from the distant nebulae and the distance to those galaxies, cosmologists soon interpreted this demonstration to mean that space itself is expanding and carrying the nebulae with it. Hence, a reluctant Einstein discarded the cosmological harmonic force.

However, not all cosmologists were willing to follow Einstein. Arthur Stanley Eddington announced that to drop the cosmological constant would knock bottom out of space! In fact, the tensor representing the geometry of space-time on the left hand side of Einstein’s field equations have a vanishing covariant derivative even if a term with a cosmological constant is included. The vanishing of the covariant derivative of the tensor representing the matter contents of the universe on the right hand side of the field equations is just the well known conservation principle of momentum and energy. Accordingly George Cunliffe McVittie claimed that the cosmological constant should be treated as an acceptable integration constant, and its value had to be obtained by astronomical observation.

As we have already mentioned, nowadays the fashion is to include the cosmological term in the energy tensor on the right hand side of Einstein’s field equations and interpret this term as a most mysterious vacuum energy. And enigmatic it is. The cosmological constant via Einstein’s equations influences space-time and every particle in the universe. But since it is a constant, nothing will influence it. This behavior violates the fundamental concept of action and reaction.

We now feel justified for our choice how to generalize Newton’s theory in an acceptable way, i.e. corresponding as much as possible to the way we generalized Einstein's theory. For small distances the inverse square term is big and govern what we observe on Earth, in the solar system, and in our galaxy. The harmonic term exists, but can safely be neglected. However on a big scale this term overwhelms the inverse square term and is the real master of what happens in the universe.

7 Horizons in Newtonian Gravity with a Repulsive Cosmological Force Included

We shall now examine the motion of particles in Newtonian gravity when a repulsive harmonic force is included. The relevant energy equation of motion for particles can
immediately be obtained integrating Newton’s second law with the force given by Newton’s ordinary inverse square law for gravitation, but now with a new harmonic term also included. However, the energy equation can as easily be obtained employing Newtonian cosmology with a repulsive force included. It is most satisfying and convincing that these 2 different procedures yield exactly the same energy equation. The first Friedmann equation for an isotropic and homogeneous universe model in Newtonian cosmology reads

\[ 8\pi G\rho = \frac{3}{a^2} \frac{\dot{a}^2}{a^2} + \frac{3}{a^2} \frac{k c^2}{a^2} - \Lambda c^2 \]  

(24)

\( a(t) \) is here a scale factor which multiplied with a co-moving radial coordinate \( r_0 \) yields the distance \( d \) of a particle from the origin, i.e. \( d = r_0 a(t) \). A dot denotes differentiation with respect to time \( t \) and \( k \) is a constant. It is immediately seen that equation (24) may be written in the following way

\[ \frac{1}{2} m \ddot{d}^2 - \frac{G \frac{4}{3} \pi \rho d^3}{d} - m \frac{1}{6} \Lambda c^2 m d^2 = \frac{1}{2} \rho_0^2 m c^2 \]  

(25)

where \( m \) is the mass of a particle.

For a dust universe we have

\[ \frac{4}{3} \pi \rho d^3 = M = \text{constant} \]  

(26)

which is the total mass contained within a sphere with radius \( d \). Hence, we have

\[ \frac{1}{2} m v^2 - \frac{G M m}{d} - m \frac{1}{6} \Lambda c^2 m d^2 = \frac{1}{2} \rho_0^2 m c^2 \]  

(27)

where \( v \) is the speed of the particle at the surface of the sphere. We recognize that equation (27) is an energy equation. We observe that the term \( -\frac{1}{2} \Lambda c^2 m d^2 \) is a kind of potential energy for a repulsive cosmological force \( F = \frac{1}{2} \Lambda c^2 m d \). The term \( -\frac{1}{2} \rho_0^2 m c^2 \) thus represents what we may call the total energy for the particle.

Next we examine what happens to a particle moving outside a spherical matter distribution when the particle is not only influenced by the Newtonian gravity force from the sphere, but also feels this cosmological force. The particle starts from the surface of the matter sphere with the speed of light \( c \) and moves radially outwards. The energy equation then reads

\[ \frac{1}{2} m v^2 = \frac{G M m}{r} + \frac{1}{6} \Lambda c^2 m v^2 + \frac{1}{2} m c^2 - \frac{G M m}{R} \frac{1}{6} \Lambda c^2 m R^2 \]  

(28)

From this equation it is seen that for a horizon to exist (where \( v = 0 \)) it must be the case that there exist positive values for the radial coordinate \( r \) such that the right hand side of (28) is negative. Remembering our discussion of equation (15) we conclude that forbidden regions with \( v^2 < 0 \) will exist only if we have

\[ \frac{1}{2} m c^2 - \frac{G M m}{R} \frac{1}{6} m c^2 \Lambda R^2 \]

\[ + \frac{1}{2} M m c^2 \left( \frac{3G M}{\Lambda c^2} \right)^{2/3} < 0 \]  

(29)

which may be written

\[ \frac{9G^2 M^2}{\Lambda^2 c^4} + \left( \frac{1}{\Lambda} - \frac{1}{3} R^3 - \frac{2G M}{\Lambda c^2 R} \right)^3 < 0 \]  

(30)

However, this condition may be written with two parameters only. We define

\[ h = \frac{G M}{\Lambda c^2 R} \]  

(31)

and
\[ \beta = R \left( \frac{2GM}{c^2} \right)^{-1} \]  

(32)

With these relations inserted condition (30) reads

\[ 9h^2 + \left( 4\beta h - \frac{1}{3} - 2h \right)^3 < 0 \]  

(33)

If condition (33) is fulfilled we find the positions of the horizons solving the following equation

\[ r^3 = \left( -\frac{2E(total)}{c^2m} \right) \frac{3}{\Lambda} r + \frac{6GM}{\Lambda c^2} = 0 \]  

(34)

where we have written

\[ E(total) = \frac{1}{2} mc^2 - \frac{GMm}{R} - \frac{1}{6} \Lambda c^2 m R^2 \]  

(35)

It should be remarked that equation (34) is both similar to and different from equation (18). We denote the smallest positive solution of equation (34) with \( r_{h1} \) and the largest solution with \( r_{h2} \). Hence, no particle can enter the interval \( <r_{h1}, r_{h2}> \).

Again several different cases exist. Remembering \( \frac{1}{2} mv^2(r = R) = \frac{1}{2} mc^2 > 0 \) it must be the case that the surface of the sphere is outside the outermost horizon or inside the innermost horizon. If \( R > r_{h2} \) in fact no horizon exist and particles can move from the surface of the sphere and reach any position far away. When horizons do exist the surface of the sphere is inside the innermost horizon if

\[ \frac{d}{dr} \left( \frac{GM}{r} + \frac{1}{6} \Lambda c^2 r^2 \right)(r = R) < 0 \]  

(36)

This condition demands

\[ h > \frac{1}{3} \]  

(37)

The physical interpretation of this condition is simply that a particle leaving the surface of the sphere will experience an inward force. We have in fact

\[ F(surface) = -\frac{GMm}{R^2} + \frac{1}{3} \Lambda c^2 R \]  

(38)

and condition (37) is immediately obtained.

Defining

\[ P(r) = \frac{1}{2} mv^2 r = \frac{1}{6} \Lambda c^2 mr^3 + E(total)r + GMm \]  

(39)

it is also immediately seen that if we have a horizon it must be the case that

\[ P(r = 0) > P(r = R) \]  

(40)

The last condition reads

\[ \beta < 1 \]  

(41)

which means that the surface is inside the Schwarzschild surface.

It is easily seen that condition (33) is fulfilled for acceptable values of \( h \) and \( \beta \). We just fix \( \beta \in <0, 1> \). For large \( h \) condition (33) is then fulfilled.

Here our conclusion is that for Newtonian gravity with a repulsive cosmological a horizon (vanishing speed for light particles) exists if and only if the following two conditions are both fulfilled:

\[ \frac{9GM^2}{\Lambda c^2} + \left( \frac{1}{\Lambda} - \frac{1}{3} R^2 - \frac{2GM}{\Lambda c^2 R} \right) < 0 \]  

and

\[ R < r_{h1} \]  

where \( r_{h1} \) is the smallest solution of the equation

\[ r^3 - \left[ -\frac{2E(total)}{c^2m} \right] \frac{3}{\Lambda} r + \frac{6GM}{\Lambda c^2} = 0 . \]

**Summary and Discussion**

It is well known the criterion to have an event horizon for the Schwarzschild line element and the condition to have a horizon for light particles leaving a massive body are exactly the same, i.e. the surface must be inside the Schwarzschild surface of the body. However,
However, we must also take into account the location of the surface of the massive gravity source. Concerning the generalized Schwarzschild metric three possibilities exist: we can have two different event horizons outside the matter sphere, one event horizon only, or no horizon. Concerning the Newtonian gravity generalized to include a repulsive cosmological constant only two possibilities exist. If the surface is outside the location of the horizons no horizon exist. If the surface is inside the innermost horizon the two horizons both exist.

Michell and Laplace assumed that light particles are massive and behave like ordinary particles when crawling upwards in a gravitational field. With our present experience we know that a light particle traveling outwards from a gravity source never slows down, but will lose energy and become redshifted moving in this geometry. We now feel safe to declare that the concurrence concerning the event horizons and the Newtonian horizons is nothing but an interesting coincidence.

The region outside the event horizon is where light and massive particles can move both inwards and outwards, while the region outside the horizon is where particles are forbidden to move.

When a repulsive cosmological term is included the necessary criteria for an event horizon to exist for the generalized Schwarzschild metric and for photons traveling outwards from a massive body to have a horizon are not identical. The first condition does not depend on the size of the mass source, but the last criterion does depend on the radius of this massive body. This second condition is rather complicated. It is, however, possible to write it with two different parameters only.

The equations which yield the event horizon $r_*$ for the Schwarzschild metric and $r_E$ for its generalized version are respectively

$$r_* - \frac{2GM}{c^2} = 0$$  \hspace{1cm} (42)

and

$$r_E^3 - \frac{3}{\Lambda} r_E + \frac{6GM}{\Lambda c^2} = 0$$  \hspace{1cm} (43)

The equations which yield the locations $r_h$ and $r_H$ for the horizons for light leaving massive bodies in the Newtonian case without and with a cosmological term are given by

$$\left[- \frac{2E_N \text{(total)}}{c^2 m}\right] r_h - \frac{2GM}{c^2} = 0$$  \hspace{1cm} (44)

and

$$r_H^3 - \left[- \frac{2E \text{(total)}}{c^2 m}\right] \frac{3}{\Lambda} r_H + \frac{6GM}{\Lambda c^2} = 0$$  \hspace{1cm} (45)

The definition of the total energy $E_N \text{(total)}$ is given by equation (35) dropping the last term.

The similarity and disparity concerning these four equations are remarkable.

However, we must also take into account the location of the surface of the massive gravity source. Concerning the generalized Schwarzschild metric three possibilities exist: we can have two different event horizons outside the matter sphere, one event horizon only, or no horizon. Concerning the Newtonian gravity generalized to include a repulsive cosmological constant only two possibilities exist. If the surface is outside the location of the horizons no horizon exist. If the surface is inside the innermost horizon the two horizons both exist.

Michell and Laplace assumed that light particles are massive and behave like ordinary particles when crawling upwards in a gravitational field. With our present experience we know that a light particle traveling outwards from a gravity source never slows down, but will lose energy and become redshifted moving in this geometry. We now feel safe to declare that the concurrence concerning the event horizons and the Newtonian horizons is nothing but an interesting coincidence.

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