Surprises with Logarithm Potential

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Abstract

The origin of logarithmic potential is investigated through a simple dimensional analysis and its physical significance has been brought out in connection with dimensional regularization of field theory. Besides, we would like to point out the bound state energy levels of logarithmic potential by uncertainty principle, phase space quantization and Hellmann-Feynman (H-F) theorem. Although the energy levels do depend on the mass of particle, however, it turns out that the energy level separation between any two levels is independent of mass term as well as Planck’s constant. We also point out the importance of this potential in 2d interacting systems in condensed matter physics.

1. Introduction

The logarithmic potential in physics forms an interesting one as it provides some unusual prediction about the system. Moreover, this potential can be used suitably to illustrate some of the important features of field theory such as dimensional regularization and renormalization. In most of our text books, this potential is not discussed at detail; although the calculations are quite simple to demonstrate some of its unique features. We have obtained the bound state energy of this logarithmic potential through uncertainty principle, phase space quantization and Hellmann-Feynman (H-F) theorem. The paper is organized as follows. With a brief discussion of the origin of this logarithmic potential through dimensional analysis, we directly go to calculate the bound state energy levels by uncertainty principle, phase space quantization and Hellmann-Feynman theorem in section 3. Finally, we give our conclusions in section 4.
2. Origin of Logarithmic potential in an electrostatic problem:

The dimensional analysis (DA) [1-3] and scaling arguments [4] form an integral part in theoretical physics to solve some important problems without doing much calculation. The dimensions of any physical quantity can be expressed in terms of three parameters mass-length-time (MLT). Every equation in science must be dimensionally homogeneous; or in other words, the left hand side (lhs) and the right hand side (rhs) of the equations must have same power of M, L and T. Whenever, there are more than one terms in an equation, it is evident that every term in such an equation must have the same dimension. This immediately indicates that the correctness of an equation can be verified by this approach. However, there exist some situations where this naïve approach may fail. In this work, we would like to discuss the implications in logarithmic function. Just like angles, exponentials and other trigonometric functions, logarithmic also falls into the category of dimensionless functions. Naturally, there will be some length scales or energy scales are built in these functions.

Let us revisit a simple problem from electrostatics where we find the logarithmic variation [5,6] of the potential with distance. Suppose an infinitely long wire (or equivalently long charged rods in 3 dimensions) is carrying a (linear) line charge density $\lambda$. We are interested to find the electric field and potential everywhere due to this charge density. Because of its obvious cylindrical symmetry, we construct cylindrical Gaussian surface with the wire as its axis and apply the Gaussian theorem to compute the electric field as

$$E = \frac{2 \lambda}{r}$$

where $r$ is the radius of the cylinder and L is its length. So, the electric field varies inversely with distance. This variation of electric field is also consistent with dimensional analysis as $[\lambda] = M^{1/2}L^{1/2}T^{-1}$; $E = M^{1/2}L^{-1/2}T^{-1}$. Now, the question is: what is the potential at any point $r$? First, we would like to sort the answer through simple dimensional analysis. But we will show below that this approach fails completely. Note that the problem has the parameter $\lambda$, therefore, the potential $\Phi(r)$ can depend on $\lambda$ and $r$ only. Thus, we can eventually write $\Phi(r) = \lambda^r r^x$. A naïve dimensional analysis reveals
that \( x = 1 \) and \( y = 0 \). Therefore, \( \Phi(r) = \lambda \) and is independent of \( r \). The potential in this case is completely dictated by the constant linear charge density \( \lambda \). But this answer is meaningless as it gives a zero electric field. But we have already noted that the electric field is not zero but varies inversely with distance. So, what’s going on here? Let us do a first principle calculation of the potential due to this line charge density. The potential can be computed as:

\[
\Phi(r) = \int_{-\infty}^{\infty} \frac{\lambda dx}{\sqrt{r^2 + x^2}} = 2\lambda \int_{0}^{\infty} \frac{dx}{\sqrt{r^2 + x^2}} \rightarrow \infty \quad (2)
\]

Moreover, by change of variable \( q = \frac{x}{r} \), a dimensionless variable, it is easy to visualize from equation (2) that

\[
\Phi(r) = 2\lambda \int_{0}^{\infty} \frac{dq}{\sqrt{q^2 + 1}} \rightarrow \infty \quad (3)
\]

The equation (2) and (3) together point out that the potential is independent of distance \( r \) and is infinite at the upper limit. Therefore, to avoid the divergence at the upper limit, we put the upper limit to \( \Lambda \) and then we can consider the infinite wire (with linear charge density) as a limiting case \( \Lambda \gg r \). Therefore, the potential \( \Phi(r, \lambda, \Lambda) \) is given by

\[
\Phi(r, \lambda, \Lambda) = 2\lambda \log \left( \frac{\Lambda}{r} \right) + 2\lambda \log \left( 1 + \sqrt{1 + \frac{r^2}{\Lambda^2}} \right) \quad (4)
\]

Thus, the finiteness of the potential is rendered by the upper-cutoff used in the first principle calculation. In the asymptotic limit \( \Lambda \gg r \), we find the potential from equation (4) as (the second term is not significant enough apart from a shift in its magnitude as seen from figure 1)

\[
\Phi(r, \lambda, \Lambda) = 2\lambda \log \left( \frac{\Lambda}{r} \right) \quad (5)
\]

Although the potential is dependent on the upper cut-off length scale \( \Lambda \), however, the physically measurable quantity electric field is not. Moreover, there is a point \( (0 < r < \infty) \), at which the potential vanishes \( (\Phi(r = \Lambda) = 0) \). This boundary condition is quite different from our usual ones where the potential vanishes only at the infinity. And, in fact, the electric field can be computed simply as
Figure 1: Schematic variation of the dimensionless potential functions as dimensionless cut-off distance $y_0$.

\[ E(r, \lambda) = \frac{2\lambda}{r} \quad (6) \]

which can be compared with equation (1). The dependence of the potential on the upper cut-off length scale $\Lambda$ can be eliminated if we can consider the difference of the potential at two distances $r$ and $a$:

\[ \Phi(r, \lambda, \Lambda) - \Phi(a, \lambda, \Lambda) = 2\lambda \log \left( \frac{a}{r} \right) \quad (7) \]

This logarithmic form of the potential can also be derived simply from scaling arguments [5] as well as dimensional regularization [6]. The arbitrary distance $a$ merely shifts the potential by a constant. Thus, it is obvious now that why the naïve arguments from
simple dimensional analysis fail in the logarithmic potential. This type feature is very
common in the discussion of renormalization group study of high energy physics \[7\] as
well as condensed matter physics \[8\]. This particular logarithmic form of the potential
does appear in computing the voltage (capacitance) difference between the coaxial
cylindrical diodes (capacitors) by solving the Laplace’s equation with appropriate
boundary conditions.

The independence of the potential with change of distance in case of infinite line charge
as evident from the naïve dimensional analysis can be demonstrated in another problem
related with Dirac-Delta function in two dimensions. The Schrödinger equation for two
dimensional attractive Dirac-Delta functions can be written as

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi - \lambda \delta^2 (r) \psi = E \psi \tag{8}
\]

In terms of scaled variables \( \bar{\lambda} = \frac{2m \lambda}{\hbar^2} \) and \( \bar{E} = \frac{2m |E|}{\hbar^2} \), the equation (8) can be written as

\[
\nabla^2 \Psi + \bar{\lambda} \delta^2 (r) \psi = \bar{E} \psi \tag{9}
\]

In two dimensions, the delta function has dimension \( M^0 L^{-2} T^0 \). Thus, \( \bar{\lambda} \) is
dimensionless while \( \bar{E} \) has the dimension of \( M^0 L^{-2} T^0 \). This indicates that we cannot
find the dependence of \( \bar{E} \) on the parameter \( \bar{\lambda} \). In other words, this particular fact in two
dimensions for Dirac-Delta potential violates the Hellmann-Feynman (H-F) theorem \[9\].

In quantum mechanics, there is no other single particle potential problem where we can
find the contradiction of powerful H-F theorem.

Another way of visualization this directly is through Fourier transformation \[5\]. Writing
\( \psi(\vec{k}) = \int \psi(\vec{r}) \exp(-i\vec{k} \cdot \vec{r}) d^2 r \), we find from equation (9) that

\[
\psi(\vec{k}) = \frac{\bar{\lambda} \psi(0)}{k^2 + \bar{E}} \tag{10}
\]

By performing the inverse Fourier transformation (integrating over \( \vec{k} \) ), we obtain the
bound state energy eigenvalue equation

\[
\frac{1}{\bar{\lambda}} = \frac{1}{4\pi^2} \int \frac{d^2 k}{k^2 + \bar{E}} = \int \frac{d^2 q}{q^2 + 1} \tag{11}
\]
In terms of change of variables $q$, we note that $\lambda$ is independent of $E$. Moreover, the integral is divergent; so introducing a momentum cut-off $\Lambda$ in the above integral with the limit $\Lambda^2 >> E$, we finally get from equation (11)

$$\frac{1}{\lambda} = -\frac{1}{4\pi} \log \left( \frac{E}{\Lambda^2} \right)$$

(12)

This immediately indicates the binding energy is given by

$$E = \Lambda^2 \exp(-4\pi / \lambda)$$

(13)

3. **Bound State Energy Levels of Logarithmic potential:**

The logarithmic potential has a branch point singularity at $r = 0$. Because of this particular type of singularity, the well-known power series method adopted for Coulomb potential ($1/r$) and harmonic oscillator potential ($r^2$) becomes ineffective in calculating the energy eigenvalues and eigenstates. However, a “shooting method” has been used successfully to obtain the exact the energy eigenvalues [10]. In this section, we would like to compute the bound state energy levels of the logarithmic potential by uncertainty principle, phase space quantization and Hellmann-Feynman (H-F) theorem. We will explicitly show although the energy levels depend on the mass term but the energy difference between any two energy levels is independent of the mass term. This is quite surprising in a typical single body potential problem in quantum mechanics.

**(a). Bound state from uncertainty principle:**

In undergraduate and post-graduate classes, although the exact results are demonstrated via Schrödinger equation; however, before the solutions, this simple method written can be illustrated as follows. The full non-relativistic Hamiltonian of this single particle in logarithmic potential can be written as

$$H = \frac{p^2}{2m} + 2\lambda \log \left( \frac{r}{a} \right)$$

(14)

There are two length scales built in to the potential one is $\lambda$ and the other is $a$ at which the potential vanishes. Using uncertainty principle, the total energy comprising of kinetic energy and the potential at length scale $b$ can be written as
\[ E(b) = E_{\text{kin}}(b) + V(b) = \frac{\hbar^2}{2mb^2} + 2\lambda \log\left(\frac{b}{a}\right) \]  \hspace{1cm} (15)

Since the dimension of \( \lambda \) is of energy dimension \( (ML^2T^{-2}) \), it is better to plot the dimensionless energy \( \hat{E} = \frac{E(b)}{2\lambda} \) as a function of \( b \) for three sets of values of \( a \) as shown in figure 1.

Figure 2: Schematic variation of the total energy as a function of distance \( b \) for three values of \( a \).
It is seen that the minimum of the energy occurs at \( b_{\min} \) which depends on \( \lambda \) but not on \( a \). The minimum energy, however, depends on both \( \lambda \) and \( a \) as

\[
E_{\min}(m, a, \lambda) = \lambda + 2\lambda \log\left(\frac{\hbar}{a\sqrt{2m\lambda}}\right) \quad (16)
\]

It is clear from equation (16) that the ground state energy essentially contains the mass term. But under the quantization condition such as \( a\sqrt{2m\lambda} \approx \hbar \), we find that the energy does not depend at all on the mass term and is given by

\[
E_n(\lambda) = \lambda(1 - 2\log n) \quad (17)
\]

This immediately indicates that

\[
\Delta E_n = E_{n+1} - E_n = 2\lambda \log\left(\frac{n}{n+1}\right) \quad (18)
\]

and the energy difference \( \Delta E_n \rightarrow 0 \) as \( n \rightarrow \infty \). The independence of the energy on mass term can be easily understood from pure dimensional analysis as the scale \( \lambda \) sets the energy in the problem. Moreover, Bohr quantization can be applied to this problem to understand the bound state energy dependence on the quantum number \( n \) as follows. If \( \vec{L} \) is the orbital angular momentum, then simple scaling analysis gives us

\[
\frac{L^2}{mr^2} \approx 2\lambda \log\left(\frac{r}{a}\right) \quad (19)
\]

Using the Bohr quantization \( L = nh \) and \( r_n \approx n^2 a \), we find the energy levels

\[
E_n(\lambda) \approx \frac{n^2 h^2}{2mr_n^2} = 2\lambda \log(n) \quad (20)
\]

The quantum mechanical nature of the problem is manifested only through the discrete quantum number \( n \) rather than Planck’s constant \( \hbar \).

**b. Bound state from Hellmann-Feynman (HF) theorem:**

To apply the Hellmann-Feynman theorem, we have to note down the virial theorem. This approach relies on computing the change in energy with respect to some parameter without explicitly knowing the wave function. Detailed accounts of this approach with several interesting problems have been discussed in this journal [11]. Although \( \vec{r} \) and \( \vec{p} \) are individually hermitean operators, however the combination \( \vec{r} \cdot \vec{p} \) is not. But we can
form the combination \( \mathbf{\hat{G}} = \mathbf{\hat{r}} \cdot \mathbf{\hat{p}} + \mathbf{\hat{p}} \cdot \mathbf{\hat{r}} \) and use it to obtain the Heisenberg’s equation for the Hamiltonian \( \mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(r) \). A simple calculation using the fundamental commutation relation \([x, p] = i\hbar\) reveals that

\[
\frac{d\langle G \rangle}{dt} = \left\langle \frac{p^2}{m} \right\rangle - \langle \mathbf{\hat{r}} \cdot \nabla V \rangle \quad (21)
\]

Now, for stationary states, the left hand side is zero, we get

\[
\left\langle \frac{p^2}{m} \right\rangle = \langle \mathbf{\hat{r}} \cdot \nabla V \rangle \quad (22)
\]

Applying equation (22) to specific logarithmic potential, we find that the expectation value of the kinetic energy in the n-th state is set by the scale \( \lambda \) as \( \langle T \rangle_n = \lambda \). This indicates that the expectation value of the full Hamiltonian is given by

\[
\langle H \rangle_n = \lambda + \langle \psi_n | V(r) | \psi_n \rangle \quad (23)
\]

Now, using Hellmann-Feynman theorem, we can calculate the variation of energy levels with respect to mass parameter. And it implies from equation (23) that

\[
\frac{\partial}{\partial m} (E_{n+1} - E_n) = 0 \quad (24)
\]

It is now evident from equation (24) that the separation between any two energy levels (need not be consecutive ones) of such potential is independent of mass.

(c). **Bound state from phase space quantization:**

In this section, we would like to compute the bound state energy from the quantization of angular momentum. However, instead of the usual quantization from WKB approximation [12, 13] for this logarithmic potential of range a,

\[
\int_0^a p(r) dr = (n - 1/4)\pi \hbar \quad (25)
\]

we proceed via the relation as indicated alternatively by the phase space integration technique[14]
\[
\int_0^\infty r(p)dp = (n+1/2)\hbar \quad (26)
\]

In the given situation, \( r = \frac{E}{2\lambda} e^{-\frac{p^2}{4\lambda}} \), we find from equation (26)

\[
ae^{\frac{E}{2\lambda}} \int_0^\infty e^{-\frac{p^2}{4\lambda}} dp = (n+1/2)\hbar \quad (27)
\]

After simple integration, the bound state energy eigen value turns out as

\[
E_n = 2\lambda \log\left[ \frac{(n+1/2)\hbar}{a\sqrt{4\pi m\lambda}} \right] \quad (28)
\]

Hence, the energy level separation between \( n \) and \( n+1 \) states is given by

\[
E_{n+1} - E_n = 2\lambda \log\left( \frac{n+3/2}{n+1/2} \right) \quad (29)
\]

This equation (29) should be compared with WKB result [9]:

\[
E_{n+1} - E_n = 2\lambda \log\left( \frac{n+3/4}{n-1/4} \right) \quad (30)
\]

In figure 3, we plot the successive energy level separation as obtained in equation (29 and (30) in units of \( \lambda \) as a function of \( n \). We have treated \( n \) as a continuous variable for the comparison of the two results obtained in the above equations. It is evident from the above figure that although there is difference in energy level separation for low quantum numbers, however, the two results match for \( n>6 \).

It is evident from equations (29) and (30) that the level separation between any two energy levels is independent of mass term \( m \) as well as Planck’s constant \( \hbar \). In other words, when a particle drops from the first excited state to the ground state, the energy of the emitted photon is \( E_1-E_0 \). However, surprisingly, the frequency of the emitted photon turns out to be independent of the mass of the particle in this logarithmic potential.
Figure 3: Comparison of successive energy level separation (measured in units of $\lambda$) as a function of $n$

The effective logarithmic interaction between quasi-particles in 2d interacting system in a strong magnetic field was first demonstrated by Laughlin [15] in connection with quantum Hall effect [16]. The many intriguing features of Laughlin wave function and its connection with Landau problem have been lucidly discussed in the literature [17].

4. Conclusions and Perspectives

To conclude, we have used the dimensional analysis as one of the easiest yet powerful tool in theoretical physics to predict the dependence of some observable quantities on the physical parameters of the bound state energy levels of logarithmic potential. The surprises encountered in the logarithmic potential have been demonstrated from different points of view. The separation between the energy levels in such a potential is shown to
be independent of the mass of the particle as well as Planck’s constant. We hope that students will benefit from the approaches of this analysis of logarithmic potential in quantum mechanics.

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References: